Chiral perturbation theory

The standard model at low energies

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Quantum chromodynamics (QCD)

Theory of strong interactions

Lagrangian of QCD

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^{\alpha}_{\mu\nu} G^{\mu\nu}_{\alpha} + \sum_f (\overline{q}_f i\gamma^\mu D_\mu q_f - m_f \overline{q}_f q_f)$$

$f = u, d, c, s, t, b$ quark flavours

$D_\mu = \partial_\mu + i g_s \frac{\lambda_\alpha}{2} G^\alpha_\mu$ covariant derivative acting in colour space

$\lambda_\alpha \quad \alpha = 1, \ldots, 8$ Gell-Mann matrices

G^α_μ gluon fields

$$G^\alpha_{\mu\nu} = \partial_\mu G^\alpha_\nu - \partial_\nu G^\alpha_\mu - g_s f_{\alpha\beta\gamma} G^\beta_\mu G^\gamma_\nu$$

Energy regions in QCD

- high energies (asymptotic freedom)
  $\longrightarrow$ perturbative QCD
- intermediate energies (resonance region)
  $\longrightarrow$ lattice, models, dispersion theory
- low energies (confinement region)
  $\longrightarrow$ chiral perturbation theory, lattice

Quark mass hierarchy in QCD

$m_u, m_d \ll m_s \ll m_c, m_b, m_t$

$\longrightarrow$ study the limit of

- $N_f = 2$ $(m_u = m_d = 0)$
  or
- $N_f = 3$ $(m_u = m_d = m_s = 0)$

massless quark flavours (chiral limit)

$$\mathcal{L}^0_{\text{QCD}} = \overline{q}_L i\gamma^\mu D_\mu q_L + \overline{q}_R i\gamma^\mu D_\mu q_R - \frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha + \mathcal{L}_{\text{heavy quarks}}$$

$$q = \begin{pmatrix} u \\ d \\ (s) \end{pmatrix}, \quad q_{R,L} = \frac{1 \pm \gamma_5}{2} q$$

QCD in the limit of massless light quarks

$\mathcal{L}^0_{\rm QCD}$ is symmetric under the global chiral transformations

$$q_R \to V_R\, q_R, \quad q_L \to V_L\, q_L, \quad V_R,\, V_L \in U(N_f)$$

Noether currents

$$V_i^\mu = \overline{q}\gamma^\mu \frac{\lambda_i}{2} q, \quad A_i^\mu = \overline{q}\gamma^\mu\gamma_5 \frac{\lambda_i}{2} q, \quad a = 1, \ldots, N^2 - 1$$

$$V_0^\mu = \overline{q}\gamma^\mu q, \quad A_0^\mu = \overline{q}\gamma^\mu\gamma_5 q$$

$U(1)_A$ anomaly: $\partial_\mu A_0^\mu = \frac{N_f \alpha_s}{4\pi} G^\alpha_{\mu\nu} \tilde{G}^{\mu\nu}_\alpha$

Chiral symmetry

$\to$ actual symmetry group of massless QCD:

$$\underbrace{SU(N_f)_L \times SU(N_f)_R}_{\text{chiral group } G} \times U(1)_V$$

$U(1)_V$: $\int d^3x\, V_0^0(x)$ counts no. of quarks minus no. of antiquarks

$\to 3V_0^\mu$ baryon number current

Quark mass term

real world:

chiral symmetry G explicitly broken by quark mass term

$$\overline{q}_L \mathcal{M} q_R + \overline{q}_R \mathcal{M} q_L$$

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & (m_s) \end{pmatrix} \quad \text{mass matrix of the light quarks}$$

Symmetries in quantum field theory

$$\partial_\mu J^\mu = 0, \quad Q = \int d^3x \, J^0(x)$$

ground state (vaccuum) $|0\rangle$, $\quad P^\mu |0\rangle = 0$

Nambu-Goldstone alternative

Two possibilities for response of vacuum to symmetry transformation

- $Q|0\rangle = 0$ Wigner-Weyl linear realization
  degenerate multiplets
  exact symmetry
- $Q|0\rangle \neq 0$ Nambu-Goldstone nonlinear realization
  massless Nambu-Goldstone bosons
  spontaneously broken symmetry

QCD vacuum

chiral symmetry spontaneously broken

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

order parameters: nonvanishing quark condensates

$$\langle 0|\overline{u}u|0\rangle = \langle 0|\overline{d}d|0\rangle = \langle 0|\overline{s}s|0\rangle \neq 0$$

vectorial subgroup $SU(N_f)_V$ remains unbroken (symmetry of the vacuum)

no. of Nambu-Goldstone bosons = no. of broken generators

- $N_f = 2$: three Nambu-Goldstone bosons $\pi^\pm, \pi^0$
- $N_f = 3$: eight Goldstone bosons $\pi^\pm, \pi^0, K^\pm, K^0, \overline{K}^0, \eta$

Spectrum of QCD in the chiral limit

- 3 (8) massless pseudoscalars
  with quantum numbers of spontaneously broken charges
- masses of ρ, p, n, ..., close to values in the real world

  relevant parameter: $\Lambda_{\rm QCD}$

  $m_{u,d} \neq 0 \to$ only few percent effect

Low-energy limit of QCD

- in the real world $m_{u,d,s} \neq 0 \longrightarrow$ chiral symmetry also explicitly broken $\longrightarrow$ pseudoscalars become massive
- but still: pseudoscalar mesons are the only asymptotic states at low energies (as long as electroweak interaction turned off)
- effects of light quark masses can be treated perturbatively

Low-energy effective quantum field theory of QCD

Chiral perturbation theory

- strategy proposed by Weinberg: construct the most general relativistic quantum field theory respecting spontaneously broken chiral symmetry and the discrete symmetries of QCD (P, C) in terms of the asymptotic fields
- systematic expansion in powers of light quark masses and momenta of the pseudoscalars
- Leutwyler, Weinberg: indeed mathematically equivalent to QCD
- price to pay: parameters (coupling constants) not further restricted by symmetries $\longrightarrow$ to be determined by lattice calculations or experiment

Geometric properties of Goldstone boson fields

Nambu-Goldstone bosons live on the coset space

$$SU(N_f)_L \times SU(N_f)_R / SU(N_f)_V$$

elements of this space can be represented by matrices $U \in SU(N_f)$ transforming as

$$U \to V_R U V_L^{-1}$$

under a chiral transformation (V_R, V_L)

possible parametrization (= choice of coordinates):
$U(\phi) = e^{i\lambda_a \phi_a / F}$ (exponential parametrization)

Construction of the effective Lagrangian

basic building block: $U(x) \in SU(N_f)$

Requirements for the effective Lagrangian:

- Lorentz invariance $\to$ only even number of derivatives possible
- chiral invariance
- P, C invariance

term without derivatives: $\langle \underbrace{UU^\dagger}_{\mathbb{1}} \rangle = \text{const.}$

$\langle \ldots \rangle$ = trace in flavour space

- no chiral invariant couplings without derivatives
- strong interaction becomes weak for small momenta
- expansion in powers of derivatives (momenta) makes sense for low energies

Lowest order Lagrangian

Nonlinear sigma model

construct term(s) with two derivatives:

$$\mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U \, \partial^\mu U^\dagger \rangle$$

- only one term at chiral order p^2 (in the chiral limit)
- parameter F cannot be further restricted by symmetry arguments
- interpretation of F: pion decay constant in the chiral limit

Physical content of $\mathcal{L}_2$

insert exponential parametrization $U = e^{i\lambda_a \phi_a / F}$ in $\mathcal{L}_2$:

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a \\ &+ \frac{1}{48F^2}\langle \lambda_a \lambda_b \lambda_c \lambda_d \rangle (\phi_a \overleftrightarrow{\partial_\mu} \phi_b)(\phi_c \overleftrightarrow{\partial^\mu} \phi_d) \\ &+ \ldots \end{aligned}$$

- correctly normalized kinetic term
- self interaction of $4, 6, \ldots$ pseudoscalar mesons
- lowest order self interaction of arbitrary even number of mesons described by single coupling parameter F

Explicit symmetry breaking and external sources

QCD in the presence of external sources:

$$\mathcal{L} = \mathcal{L}^0_{\text{QCD}} + \overline{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \overline{q}(s - i\gamma_5 p)q$$

v_μ, a_μ, s, p are external (matrix-valued) sources (classical fields)

- vector source v_μ
- axial-vector source a_μ
- scalar source s
- pseudoscalar source p

Advantages of external field method

Gasser, Leutwyler

- external photon and $W^\pm$ fields automatically included with

$$\begin{aligned}
r_\mu &= v_\mu + a_\mu = -eQA_\mu \\
l_\mu &= v_\mu - a_\mu = -eQA_\mu - \frac{e}{\sqrt{2}\sin\theta_W}(W^\pm_\mu T_+ + h.c.) \\
Q &= \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix}, \quad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

- Green functions of quark currents can be obtained from the generating functional

$$e^{iW[v,a,s,p]} = \int [dq\, d\overline{q}\, dG] e^{i\mathcal{L}}$$

External field method (ctd.)

- explicit chiral symmetry breaking through the light quark masses obtained by evaluting the functional derivatives at

$$s = \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

- axial vector currents are interpolating fields for the pseudoscalars

$$\langle 0 | A_a^\mu(x) | \phi_b(p) \rangle = i \delta_{ab} F_b e^{-ip\cdot x} p^\mu$$

External field method (ctd.)

- Green functions of axial-vector currents

$$\langle 0 | \mathrm{T}\, A\, A\, A\, A | 0 \rangle_c = \frac{\delta}{\delta a} \frac{\delta}{\delta a} \frac{\delta}{\delta a} \frac{\delta}{\delta a} W[v, a, s, p] \Big|_{v=a=p=0,\, s=\mathcal{M}}$$

$\rightarrow$ transition matrix elements of meson-meson scattering

$$\langle\, \phi\, \phi\, \mathrm{out} |\, \phi\, \phi\, \mathrm{in} \rangle$$

- generating functional $W[v, a, s, p]$ can be calculated in a manifestly chiral invariant way till the very end before taking the appropriate functional derivatives for the actual Green functions

Local chiral symmetry

$\mathcal{L}$ exhibits a local $SU(N_f)_L \times SU(N_f)_R$ symmetry:

$$
\begin{aligned}
q_{L,R}(x) &\rightarrow V_{L,R}(x)\, q_{L,R}(x) \\
r_\mu(x) &\rightarrow V_R(x)\, r_\mu(x)\, V_R^\dagger(x) + i V_R(x)\, \partial_\mu V_R^\dagger(x) \\
\ell_\mu(x) &\rightarrow V_L(x)\, \ell_\mu(x) V_L^\dagger(x) + i V_L(x)\, \partial_\mu V_L^\dagger(x) \\
(s+ip)(x) &\rightarrow V_R(x)\,(s+ip)(x)\, V_L^\dagger(x)
\end{aligned}
$$

Ward identities

infinitesimal chiral tranformation:

$$
\begin{aligned}
V_R(x) &= \mathbb{1} + i\alpha(x) + i\beta(x) + \ldots \\
V_L(x) &= \mathbb{1} + i\alpha(x) - i\beta(x) + \ldots
\end{aligned}
$$

corresponding gauge transformation of the external fields:

$$
\begin{aligned}
\delta v_\mu &= \partial_\mu \alpha + i[\alpha, v_\mu] + i[\beta, a_\mu] \\
\delta a_\mu &= \partial_\mu \beta + i[\alpha, a_\mu] + i[\beta, v_\mu] \\
\delta s &= i[\alpha, s] - \{\beta, p\} \\
\delta p &= i[\alpha, p] + \{\beta, s\}
\end{aligned}
$$

Chiral anomaly

response of the QCD functional:

$$
\begin{aligned}
\delta W &= -\frac{N_c}{(4\pi)^2}\int d^4x\,\varepsilon^{\alpha\beta\mu\nu}\Big\langle \beta\big(v_{\alpha\beta}v_{\mu\nu} + \frac{4}{3}\nabla_\alpha a_\beta \nabla_\mu a_\nu \\
&\quad + \frac{2i}{3}\big\{v_{\alpha\beta}, a_\mu a_\nu\big\} + \frac{8i}{3} a_\mu v_{\alpha\beta} a_\nu + \frac{4}{3} a_\alpha a_\beta a_\mu a_\nu\big)\Big\rangle \\
v_{\alpha\beta} &= \partial_\alpha v_\beta - \partial_\beta v_\alpha - i[v_\alpha, v_\beta] \\
\nabla_\alpha a_\beta &= \partial_\alpha a_\beta - i[v_\alpha, a_\beta]
\end{aligned}
$$

general solution: $W = W_{\text{inv}} + W_{\text{WZW}}$

- general chiral invariant functional W_{inv}
- Wess-Zumino-Witten functional W_{WZW}

Effective field theory with external sources

$$
e^{iW[v,a,s,p]} = \int [dG\, dq\, d\overline{q}] e^{i\int d^4x\,\mathcal{L}} = \int [dU] e^{i\int d^4x\,\mathcal{L}_{\text{eff}}}
$$

expansion of $\mathcal{L}_{\text{eff}}$ in powers of derivatives and external fields:

$$
\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots
$$

covariant derivative:

$$
\partial_\mu U \rightarrow D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu
$$

nonabelian field strength tensors:

$$
\begin{aligned}
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
F_L^{\mu\nu} &= \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i[\ell^\mu, \ell^\nu]
\end{aligned}
$$

Chiral counting

U	$\mathcal{O}(p^0)$
$D_\mu,\ v_\mu,\ a_\mu$	$\mathcal{O}(p)$
$s,\ p,\ F^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

Lowest order effective Lagrangian

$$\mathcal{L}_2 = \frac{F^2}{4}\langle D_\mu U\, D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle, \quad \chi = 2B(s+ip)$$

two free constants at $\mathcal{O}(p^2)$: F, B

- $F =$ pion decay constant to lowest order: $F = F_\pi\big(1+\mathcal{O}(p^2)\big)$ ($F_\pi = 92.2\,\mathrm{MeV}$)
- B is related to the quark condensate:

$$\left.\frac{\delta W}{\delta s_{ij}}\right|_{v=a=s=p=0} = \langle 0|\overline{q}_i q_j|0\rangle = F^2 B\delta_{ij} + \dots$$

Meson masses to lowest order

- $s \to \mathcal{M}$ in $\mathcal{L}_2$
- insert exponential parametrization $U = \exp(i\sqrt{2}\phi/F)$ in $\mathcal{L}_2$ and extract terms bilinear in ϕ

ϕ expressed in terms of physical fields (in the isopin limit $m_u = m_d$):

$$\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

Meson masses to lowest order (ctd.)

$$\begin{aligned} M^2_{\pi^\pm} &= B(m_u + m_d) \\ M^2_{\pi^0} &= B(m_u + m_d) \\ M^2_{K^\pm} &= B(m_s + m_u) \\ M^2_{K^0} &= B(m_s + m_d) \\ M^2_{\eta} &= \frac{4}{3}B(m_s + m_u + m_d) \end{aligned}$$

- $M_{K^0} > M_{K^+} \;\to\; m_d > m_u$
- GMO relation $3(M^2_\eta - M^2_\pi) = 4(M^2_K - M^2_\pi)$

Next-to-leading order

- one-loop graphs with vertices generated by $\mathcal{L}_2$
- tree graphs generated by $\mathcal{L}_4$ (Gasser-Leutwyler)
- contributions from Wess-Zumino-Witten functional

typical suppression factor $(M/4\pi F)^2$ in amplitudes

- $(M_\pi/4\pi F)^2 \sim$ few percent in $SU(2)_L \times SU(2)_R$
- $(M_K/4\pi F)^2 \sim$ 20 percent in $SU(3)_L \times SU(3)_R$

Gasser-Leutwyler Lagrangian

$$\begin{aligned}
\mathcal{L}_4 \;=\;& L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + \chi U^\dagger)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&-iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle \\
&+L_{11}\langle F_R^{\mu\nu} F_{R\mu\nu}\rangle\langle F_L^{\mu\nu} F_{L\mu\nu}\rangle + L_{12}\langle \chi^\dagger \chi\rangle
\end{aligned}$$

UV-divergences of one-loop graphs absorbed by divergent parts of low-energy couplings $L_i = L_i^{\text{div}} + L_i^{\text{finite}}$

L_i^{finite} determined from

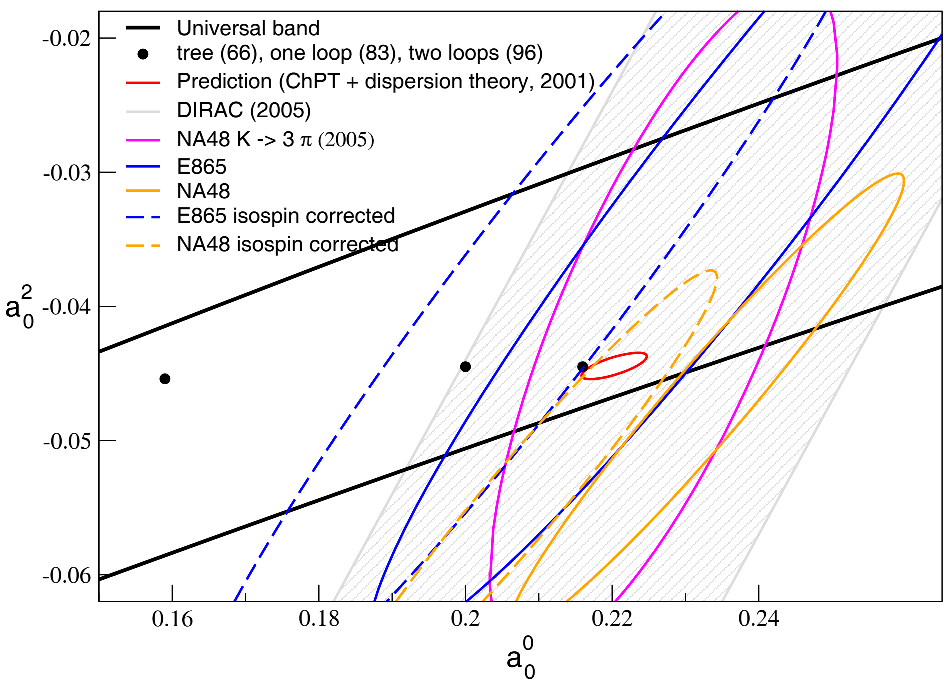
- experiment
- lattice calculations
- model calculations

$\pi\pi$ scattering

s-wave scattering lengths:

$$a_0^0 = \underbrace{\frac{7M_\pi^2}{32\pi F_\pi^2}}_{\text{tree}} = 0.159 \rightarrow \underbrace{0.200}_{\text{one-loop}} \rightarrow \underbrace{0.216}_{\text{two-loop}}$$

$$a_0^2 = \underbrace{-\frac{M_\pi^2}{16\pi F_\pi^2}}_{\text{tree}} = -0.0454 \rightarrow \underbrace{-0.0445}_{\text{one-loop}} \rightarrow \underbrace{-0.0445}_{\text{two-loop}}$$



from: G. Colangelo, J. Gasser, A. Rusetsky, EPJC **59** (2009) 777

Extensions of chiral perturbation theory

- Nonleptonic weak interactions
- Chiral perturbation theory with virtual photons and leptons
- Chiral perturbation theory with baryons
- Hadronic atoms

Conclusions

- Chiral perturbation theory is the low energy effective field theory of the standard model.
- It allows precision physics at low energies in several areas.
- Further progress in the field is expected from lattice determinations of low-energy constants.